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On the Approximate Solution of D'Alembert Type Equation Originating from Number Theory

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Abstract: We solve the functional equation

$$E(\alpha):\ f(x_1x_2+\alpha y_1y_2, x_1y_2+x_2y_1)+f(x_1x_2-\alpha y_1y_2, x_2y_1-x_1y_2)=2f(x_1,y_1)f(x_2,y_2),$$

where $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{C}$ and α is a real parameter, on the monoid $\mathbb{R}^2$. Also we investigate the stability of this equation in the following setting:

$$|f(x_1x_2+\alpha y_1y_2, x_1y_2+x_2y_1)+f(x_1x_2-\alpha y_1y_2, x_2y_1-x_1y_2)-2f(x_1,y_1)f(x_2,y_2)| \\ \le \min\{\varphi(x_1),\psi(y_1),\phi(x_2),\zeta(y_2)\}.$$

From this result, we obtain the superstability of this equation.

Key words: D'Alembert functional equation, monoid $\mathbb{R}^2$, multiplicative function, stability, superstability.

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1. Introduction

For any $\alpha\in\mathbb{R}$, Berrone and Dieulefait [5] equipped $\mathbb{R}^2$ with the multiplication rule $\cdot_\alpha$, defined by

$$(x_1,y_1)\cdot_\alpha(x_2,y_2)=(x_1x_2+\alpha y_1y_2, x_1y_2+x_2y_1),\quad (x_1,y_1),\,(x_2,y_2)\in\mathbb{R}^2.$$

For $\alpha=-1$, the multiplication is the usual product of complex numbers in $\mathbb{C}=\mathbb{R}^2$. The rule makes $\mathbb{R}^2$ into a commutative monoid with neutral element $(1,0)$ and $\sigma(x,y)=(x,-y)$ (complex conjugation) as an involution.

Berrone and Dieulefait [5, Theorem 1] studied the homomorphisms $m:(\mathbb{R}^2,\cdot_\alpha)\longrightarrow(\mathbb{R},.)$, i.e., the multiplicative, real-valued functions on the monoid $(\mathbb{R}^2,\cdot_\alpha)$. We extend their investigations by finding the bigger set of all multiplicative, complex-valued functions $M:(\mathbb{R}^2,\cdot_\alpha)\longrightarrow(\mathbb{C},.)$. Combining this information with Davison's work [9] about D'Alembert's functional equation on monoids, we obtain an explicit description of the solutions $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ of D'Alembert's functional equation

$$E(\alpha) : \ f(a \cdot_\alpha b) + f(a \cdot_\alpha \sigma(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2,$$

on the monoid $(\mathbb{R}^2, \cdot_\alpha)$. The description falls into three different cases, according to whether $\alpha > 0$ or $\alpha < 0$. The equation $E(\alpha)$ is a common generalization of many functional equations of type D'Alembert

$$f(ab) + f(a\sigma(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2 \tag{1.1}$$

on the monoid $\mathbb{R}^2$, like, e.g.,

1) If $\alpha = 0$,

$$E(0) : f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, x_2y_1 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Setting $x_1 = x_2 = 1$ and $F(y) = f(1, y)$ for any $y \in \mathbb{R}$ respectively $y_1 = y_2 = 0$ and $m(x) = f(x, 0)$ for any $x \in \mathbb{R}$ in $E(0)$, we get the classical D'Alembert functional equation

$$F(y_1 + y_2) + F(y_1 - y_2) = 2F(y_1)F(y_2), \quad y_1, y_2 \in \mathbb{R} \tag{1.2}$$

on $\mathbb{R}$ (see [1], [4], [15] and [23]) respectively the classical Cauchy equation

$$m(x_1x_2) = m(x_1)m(x_2), \quad x_1, x_2 \in \mathbb{R} \tag{1.3}$$

on $\mathbb{R}$. We call m a multiplicative function on $\mathbb{R}$ (see[1]).

2) If $\alpha = -1$,

$$\begin{aligned} E(-1) : f(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 + y_1y_2, x_2y_1 - x_1y_2) \\ = 2f(x_1, y_1)f(x_2, y_2), \end{aligned}$$

$(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. The equation $E(-1)$ is in connection with the identity

$$\begin{aligned} (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 + (x_1x_2 + y_1y_2)^2 + (x_2y_1 - x_1y_2)^2 \\ = 2(x_1^2 + y_1^2)(x_2^2 + y_2^2) \end{aligned} \tag{1.4}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

3) If $\alpha \neq 1$ is a square free integer and $\mathbb{Q}(\sqrt{\alpha}) = \{x + y\sqrt{\alpha} :\ x, y \in \mathbb{Q}\}$ is the quadratic monoid equipped with the multiplicative rule

$$(x_1 + y_1\sqrt{\alpha})(x_2 + y_2\sqrt{\alpha}) = (x_1x_2 + \alpha y_1y_2) + (x_1y_2 + x_1y_1)\sqrt{\alpha}, \qquad (1.5)$$

then $E(\alpha)$ reduces to D'Alembert functional equation (1.1) on the monoid $\mathbb{Q}(\sqrt{\alpha})$. In [9] Davison solved the D'Alembert functional equation with involution on a monoid A: any solution $f : A \longrightarrow \mathbb{C}$ has the general form $f = \frac{M + M\circ\sigma}{2}$, where $M : A \longrightarrow\longrightarrow \mathbb{C}$ is a multiplicative function.

In 1940, Ulam [22] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Question 1.1. Let $(G_1, *)$ be a group and let $(G_1, \diamond, d)$ be a metric group with the metric d. Given $\varepsilon > 0$, does there exist $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \delta(\varepsilon)$ for all $x \in G_1$?

In 1941, Hyers [12] answered this question for the case where G_1 and G_2 are Banach spaces. In 1978, Rassias [20] provided a generalization of Hyer's theorem which allows the Cauchy difference to be unbounded. The interested reader may refer to the book by Hyers, Isac, Rassias [13] for an in depth account on the subject of stability of functional equations. In 1982, Rassias [19] solved the Ulam problem by involving a product of powers of norms. Since then, the stability problems of various functional equations ha been investigated by many authors (see [10], [11] and [14]). In [3] and [7] Baker et al. and Bourgin respectively, introduced the notion that by now is frequently referred to as superstability or Baker's stability: if a function f satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. The superstability of D'Alembert's functional equation $f(x + y) + f(x - y) = 2f(x)f(y)$ was investigated by Baker [4] and Cholewa [8]. Badora and Ger [2], and Kim ([16], [17] and [18]) proved its superstability under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$. In a previuos work, Bouikhalene et al. [6] investigated the superstability of the cosine functional equation on the Heisenberg group. Following this investigation we study the superstability of the functional equation $E(\alpha)$ on the monoid $(\mathbb{R}^2, \cdot_\alpha)$. Also we say that a function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is of approximate a cosine type function, if there is $\delta > 0$ such that

$$|f(a \cdot_\alpha b) + f(a \cdot_\alpha i(b)) - 2f(a)f(b)| < \delta, \quad a, b \in \mathbb{R}^2. \tag{1.6}$$

In the case where $\delta = 0$, f satisfies the functional equation $E(\alpha)$. We call f a cosine type function on $\mathbb{R}^2$. The paper is organized as follows: In the first section after this introduction we solve the functional equation $E(\alpha)$. In the second section we study the superstability equation $E(\alpha)$.

2. Solution of equation $E(\alpha)$

According to [9] we drive the following lemma.

Lemma 2.1. *The solution $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ of $E(\alpha)$ is of the form*

$$f = \frac{M + M \circ \sigma}{2},$$

where $M : (\mathbb{R}^2, \cdot_\alpha) \longrightarrow (\mathbb{C}, \cdot)$ is a multiplicative function.

By extending Berrone-Dieulefait's result [5] to complex-valued multiplicative functions, we get the following lemmas.

Lemma 2.2. *The multiplicative functions $M : (\mathbb{R}^2, \cdot_1) \longrightarrow (\mathbb{C}, \cdot)$ are the functions*

$$M(x, y) = m_1(x + y)m_2(x - y), \quad x, y \in \mathbb{R},$$

where $m_1, m_2 : \mathbb{R} \longrightarrow \mathbb{C}$ are multiplicative functions.

Lemma 2.3. *The multiplicative functions $M : (\mathbb{R}^2, \cdot_0) \longrightarrow (\mathbb{C}, \cdot)$ are the trivial function $M = 1$ and $M(0, y) = 0$ for any $y \in \mathbb{R}$ and $M(x, y) = m(x)\gamma(\frac{y}{x})$ for any $(x, y) \in \mathbb{R}^2$, with $x \neq 0$, where $m : \mathbb{R} \longrightarrow \mathbb{C}$ is a multiplicative function and $\gamma : (\mathbb{R}, +) \longrightarrow \mathbb{C}$ is an arbitrary character.*

Lemma 2.4. *The multiplicative functions $M : (\mathbb{C}, \cdot_{-1}) \longrightarrow (\mathbb{C}, \cdot)$ are the trivial functions $M = 0$ and $M = 1$ and*

$$M(z) = \begin{cases} \widetilde{m}(|z|)\Gamma(\exp(i\theta)), & \text{for } z = |z|\exp(i\theta) \neq 0 \\ 0, & \text{for } z = 0. \end{cases}$$

where $\widetilde{m} : (\mathbb{R}^+, \cdot) \longrightarrow \mathbb{C}^*$ and $\Gamma : \{\exp(i\theta),\ \theta \in \mathbb{R}\} \longrightarrow \mathbb{C}^*$ are arbitrary characters.

Proof. When $\alpha = -1$, the multiplicative rule $\cdot_{-1}$ becomes the usual product numbers in $\mathbb{C}$. By using the polar decomposition $z = |z|\exp(i\theta)$ for any $z \in \mathbb{C}^*$ where $\theta = \arg(z)$, we get

$$M(|z_1||z_2|) = M(|z_1|)M(|z_2|), \quad z_1, z_2 \in \mathbb{C}^* \tag{2.1}$$

and

$$M(\exp(i(\theta_1+\theta_2))) = M(\exp(i\theta_1))M(\exp(i\theta_2)), \quad \theta_1, \theta_2 \in \mathbb{R}. \tag{2.2}$$

By letting $\widetilde{m}(|z|) = M(|z|)$, for any $z \in \mathbb{C}^*$, and $\Gamma(\exp(i\theta)) = M(\exp(i\theta))$ for any $\theta \in \mathbb{R}$ it follows that $\widetilde{m} : (\mathbb{R}^+, \cdot) \longrightarrow \mathbb{C}^*$ and $\Gamma : \{\exp(i\theta),\ \theta \in \mathbb{R}\} \longrightarrow \mathbb{C}^*$ are characters. If $z = 0$, we set $M(z) = 0$. ∎

In the next corollary we give the set of all multiplicative complex-valued functions $M : (\mathbb{R}^2, \cdot_\alpha) \longrightarrow \mathbb{C}$.

Corollary 2.5. *The multiplicative functions $M : (\mathbb{R}^2, \cdot_\alpha) \longrightarrow (\mathbb{C}, \cdot)$ are given by the following list:*

I) *If $\alpha > 0$, then*

$$M(x,y) = m_1(x + y\sqrt{\alpha})m_2(x - y\sqrt{\alpha}), \quad (x,y) \in \mathbb{R}^2.$$

II) *If $\alpha = 0$, then*

a) $M(x,y) = 1$, *for any* $(x,y) \in \mathbb{R}^2$.
b) $M(0,y) = 0$, *for any* $y \in \mathbb{R}$.
c) $M(x,y) = m(x)\gamma(\frac{y}{x})$, *for any* $(x,y) \in \mathbb{R}^2$ *with* $x \neq 0$.

III) *If $\alpha < 0$, then*

a) $M(x,y) = 0$, *for any* $(x,y) \in \mathbb{R}^2$.
b) $M(x,y) = 1$, *for any* $(x,y) \in \mathbb{R}^2$.
c) $M(x,y) = \begin{cases} \widetilde{m}(\sqrt{x^2 - \alpha y^2})\Gamma(\arg(x+iy)), & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0). \end{cases}$

where $m_1, m_2, m : \mathbb{R} \longrightarrow \mathbb{C}$ are multiplicative functions, and $\widetilde{m} : (\mathbb{R}^+, \cdot) \longrightarrow \mathbb{C}^*$, $\Gamma : \{\exp(i\theta),\ \theta \in \mathbb{R}\} \longrightarrow \mathbb{C}^*$ and $\gamma : (\mathbb{R}, +) \longrightarrow \mathbb{C}$ are arbitrary characters.

The next theorem is the main result of this section.

Theorem 2.6. *The set of solutions of the functional equation $E(\alpha)$ consists of the following three cases:*

A) *If $\alpha > 0$, then*

$$f(x,y) = \frac{m_1(x)m_2(y)}{2}\big\{m_1(y\sqrt{\alpha})m_2(-y\sqrt{\alpha}) + m_1(-y\sqrt{\alpha})m_2(y\sqrt{\alpha})\big\},$$

for any $(x,y) \in \mathbb{R}^2$.

B) *If $\alpha = 0$, then*

a) $f(x,y) = 1$, *for any* $(x,y) \in \mathbb{R}^2$.
b) $f(0,y) = 0$, *for any* $y \in \mathbb{R}$.
c) $f(x,y) = \frac{m(x)}{2}\big\{\gamma(\frac{y}{x}) + \gamma(\frac{-y}{x}),\ (x,y) \in \mathbb{R}^2,\ x \neq 0.\big\}$

C) *If $\alpha < 0$, then $f(0,0) = 0$ and*

$$f(x,y) = \frac{\widetilde{m}\big(\sqrt{x^2 - \alpha y^2}\big)}{2}\big\{\Gamma(\arg(x+iy)),\ (x,y) \in \mathbb{R}^2\backslash(0,0)\big\},$$

where $m_1, m_2, m : \mathbb{R} \longrightarrow \mathbb{C}$ are multiplicative functions, and $\widetilde{m} : (\mathbb{R}^+, \cdot) \longrightarrow \mathbb{C}^*$, $\Gamma : \{\exp(i\theta),\ \theta \in \mathbb{R}\} \longrightarrow \mathbb{C}^*$ and $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ are arbitrary characters.

Proof. According to Lemma 2.1 and Corollary 2.5 we get the proof of theorem. ∎

3. Superstability of equation $E(\alpha)$

In the next theorem we establish the stability of $E(\alpha)$.

Theorem 3.1. *Let $\varphi, \psi, \phi, \zeta : \mathbb{R} \longrightarrow [0, +\infty[$ be functions and let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a function such that*

$$\begin{aligned}\big|f(x_1x_2+\alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2)\\ - 2f(x_1,y_1)f(x_2,y_2)\big| \leq \min\{\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)\}\end{aligned} \tag{3.1}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *and* α *is a real parameter. Then either* f *is bounded or* f *satisfies the functional equation*

$$E(\alpha) : f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) \\ = 2f(x_1, y_1)f(x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Proof. For all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and α a real parameter we get from the inequality (3.1) that

$$\begin{aligned} \big| f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) \\ - 2f(x_1, y_1)f(x_2, y_2)\big| \\ \leq \varphi(x_1) \text{ or } \psi(y_1). \end{aligned} \tag{3.2}$$

Since f is unbounded then we can choose a sequence $(x_n, y_n)_{n\geq 3}$ in $\mathbb{R}^2$ such that $f(x_n, y_n) \neq 0$ and $\lim_{n\to+\infty} |f(x_n, y_n)| = +\infty$. Taking $(x_2, y_2) = (x_n, y_n)$ in (3.2) we obtain

$$\begin{aligned} \big| f(x_1x_n + \alpha y_1y_n, x_1y_n + x_ny_1) + f(x_1x_n - \alpha y_1y_n, x_ny_1 - x_1y_n) \\ - 2f(x_1, y_1)f(x_n, y_n)\big| \\ \leq \varphi(x_1) \text{ or } \psi(y_1) \end{aligned}$$

and

$$\left| \frac{f(x_1x_n + \alpha y_1y_n, x_1y_n + x_ny_1) + f(x_1x_n - \alpha y_1y_n, x_ny_1 - x_1y_n)}{2f(x_n, y_n)} - f(x_1, y_1) \right| \\ \leq \frac{\varphi(x_1)}{2|f(x_n, y_n)|} \text{ or } \frac{\psi(y_1)}{2|f(x_n, y_n)|}.$$

That is we get

$$\begin{aligned} &f(x_1, y_1) \\ &= \lim_{n\to+\infty} \frac{f(x_1x_n + \alpha y_1y_n, x_1y_n + x_ny_1) + f(x_1x_n - \alpha y_1y_n, x_ny_1 - x_1y_n)}{2f(x_n, y_n)}. \end{aligned} \tag{3.3}$$

Setting $X_n = x_2x_n + \alpha y_2y_n,\ Y_n = x_2y_n + x_ny_2,\ \widetilde{X}_n = x_2x_n - \alpha y_2y_n,\ \widetilde{Y}_n = x_2y_n - x_ny_2$. For any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ it follows that

$$
\begin{aligned}
&\big|f\big((x_1x_2+\alpha y_1y_2)x_n+\alpha(x_1y_2+x_2y_1)y_n,\\
&\qquad\qquad\qquad\qquad (x_1x_2+\alpha y_1y_2)y_n+x_n(x_1y_2+x_2y_1)\big)\\
&\quad+f\big((x_1x_2+\alpha y_1y_2)x_n-\alpha(x_1y_2+x_2y_1)y_n,\\
&\qquad\qquad\qquad\qquad x_n(x_1y_2+x_2y_1)-(x_1x_2+\alpha y_1y_2)y_n\big)\\
&\quad-2f(x_1,y_1)f(x_2x_n+\alpha y_2y_n,x_2y_n+x_ny_2)\\
&\quad+f\big((x_1x_2-\alpha y_1y_2)x_n+\alpha(x_2y_1-x_1y_2)y_n,\\
&\qquad\qquad\qquad\qquad (x_1x_2-\alpha y_1y_2)y_n+x_n(x_2y_1-x_1y_2\big)\\
&\quad+f\big((x_1x_2-\alpha y_1y_2)x_n-\alpha(x_2y_1-x_1y_2)y_n,\\
&\qquad\qquad\qquad\qquad x_n(x_2y_1-x_1y_2)-(x_1x_2-\alpha y_1y_2)y_n\big)\\
&\quad-2f(x_1,y_1)f(x_2x_n-\alpha y_2y_n,x_2y_n-x_ny_2)\big|\\
&\le\big|f\big((x_1x_2+\alpha y_1y_2)x_n+\alpha(x_1y_2+x_2y_1)y_n,\\
&\qquad\qquad\qquad\qquad (x_1x_2+\alpha y_1y_2)y_n+x_n(x_1y_2+x_2y_1)\big)\\
&\quad+f\big((x_1x_2-\alpha y_1y_2)x_n-\alpha(x_2y_1-x_1y_2)y_n,\\
&\qquad\qquad\qquad\qquad x_n(x_2y_1-x_1y_2)-(x_1x_2-\alpha y_1y_2)y_n\big)\\
&\quad-2f(x_1,y_1)f(x_2x_n+\alpha y_2y_n,x_2y_n+x_ny_2)\big|\\
&\quad+\big|f\big((x_1x_2-\alpha y_1y_2)x_n+\alpha(x_2y_1-x_1y_2)y_n,\\
&\qquad\qquad\qquad\qquad (x_1x_2-\alpha y_1y_2)y_n+x_n(x_2y_1-x_1y_2)\big)\\
&\quad+f\big((x_1x_2+\alpha y_1y_2)x_n-\alpha(x_1y_2+x_2y_1)y_n,\\
&\qquad\qquad\qquad\qquad x_n(x_1y_2+x_2y_1)-(x_1x_2+\alpha y_1y_2)y_n\big)\\
&\quad-2f(x_1,y_1)f(x_2x_n-\alpha y_2y_n,x_2y_n-x_ny_2)\big|\\
&=\big|f(x_1X_n+\alpha y_1Y_n,x_1Y_n+X_ny_1)+f(x_1X_n-\alpha y_1Y_n,X_ny_1-x_1Y_n)\\
&\qquad\qquad\qquad\qquad -2f(x_1,y_1)f(X_n,Y_n)\big|\\
&\quad+\big|f(x_1\widetilde{X}_n+\alpha y_1\widetilde{Y}_n,x_1\widetilde{Y}_n+\widetilde{X}_ny_1)+f(x_1\widetilde{X}_n-\alpha y_1\widetilde{Y}_n,\widetilde{X}_ny_1-x_1\widetilde{Y}_n)\\
&\qquad\qquad\qquad\qquad -2f(x_1,y_1)f(\widetilde{X}_n,\widetilde{Y}_n)\big|\\
&\le 2\varphi(x_1)\text{ or }2\psi(y_1).
\end{aligned}
$$

So that

$$\begin{aligned}
&\left| \frac{f\big((x_1x_2+\alpha y_1y_2)x_n+\alpha(x_1y_2+x_2y_1)y_n,\ (x_1x_2+\alpha y_1y_2)y_n+x_n(x_1y_2+x_2y_1)\big)}{f(x_n,y_n)} \right.\\
&+\frac{f\big((x_1x_2+\alpha y_1y_2)x_n-\alpha(x_1y_2+x_2y_1)y_n,\ x_n(x_1y_2+x_2y_1)-(x_1x_2+\alpha y_1y_2)y_n\big)}{f(x_n,y_n)}\\
&+\frac{f\big((x_1x_2-\alpha y_1y_2)x_n+\alpha(x_2y_1-x_1y_2)y_n,\ x_n(x_2y_1-x_1y_2)+(x_1x_2-\alpha y_1y_2)y_n\big)}{f(x_n,y_n)}\\
&+\frac{f\big((x_1x_2-\alpha y_1y_2)x_n-\alpha(x_2y_1-x_1y_2)y_n,\ x_n(x_2y_1-x_1y_2)-(x_1x_2-\alpha y_1y_2)y_n\big)}{f(x_n,y_n)}\\
&\left.-2f(x_1,y_1)\left\{\frac{f(x_2x_n+\alpha y_2y_n,x_2y_n+x_ny_2)+f(x_2x_n-\alpha y_2y_n,x_2y_n-x_ny_2)}{f(x_n,y_n)}\right\}\right|\\
&\le 2\frac{\varphi(x_1)}{|f(x_n,y_n)|} \text{ or } 2\frac{\psi(y_1)}{|f(x_n,y_n)|}.
\end{aligned}$$

for any $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$. Since $|f(x_n,y_n)|\longrightarrow+\infty$ as $n\longrightarrow+\infty$ we get that f satisfies $E(\alpha)$. ∎

By letting $\min\{\varphi(x_1),\psi(y_1),\phi(x_2),\zeta(y_2)\}=\delta$ we get the Baker's stability ([3], [4]) for the functional equation $E(\alpha)$.

Corollary 3.2. *Let $\delta>0$ and let $f:\mathbb{R}^2\longrightarrow\mathbb{C}$ be a function such that*

$$\begin{aligned}
|f(x_1x_2+\alpha y_1y_2,x_1y_2+x_2y_1)+f(x_1x_2-\alpha y_1y_2,x_2y_1-x_1y_2)&\\
-2f(x_1,y_1)f(x_2,y_2)|\le\delta&
\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ and α is a real parameter. Then either f is bounded and $|f(x,y)|\le\frac{1+\sqrt{1+2\delta}}{2}$ for all $(x,y)\in\mathbb{R}^2$ or f satisfies the functional equation $E(\alpha)$.

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